第11章 应力状态和强度理论

§11.2 平面应力状态应力分析的解析法

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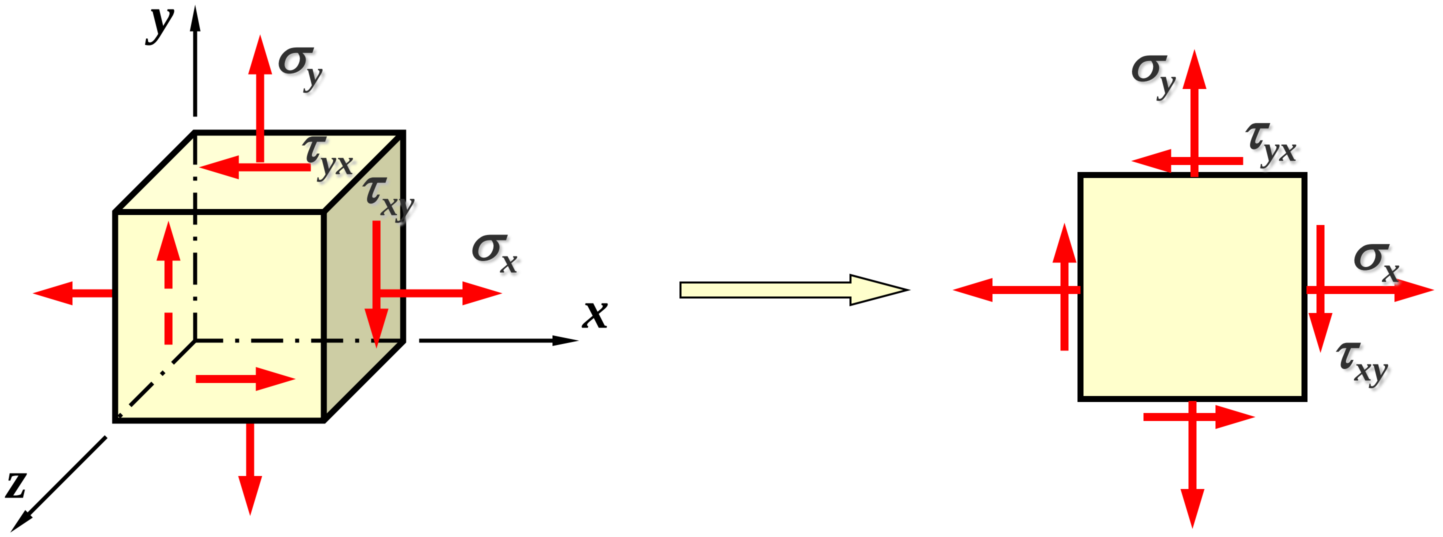


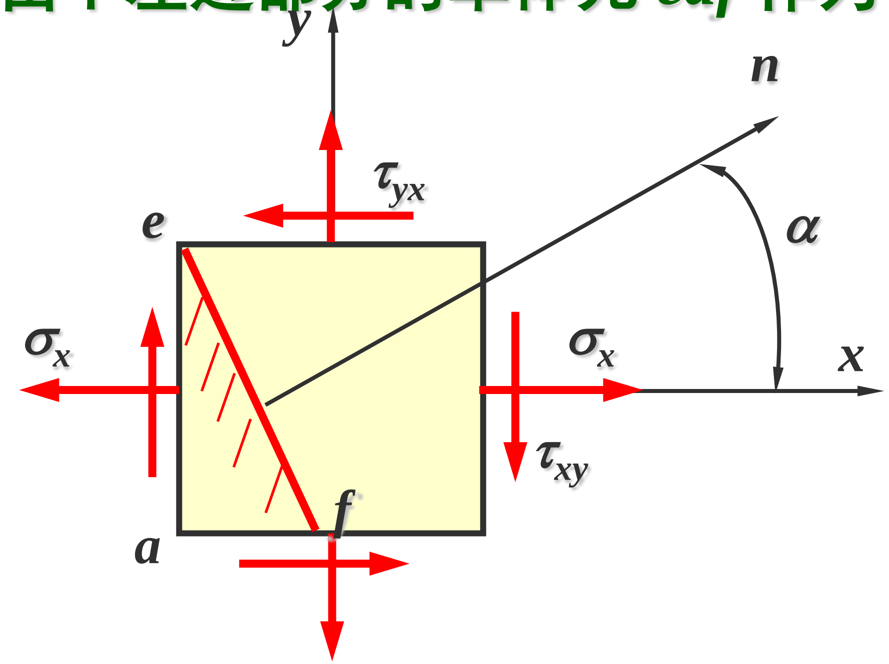
二向应力状态的普遍形式如图所示，单元体上有σ_x，τ_{xy}和σ_y，τ_{yx}

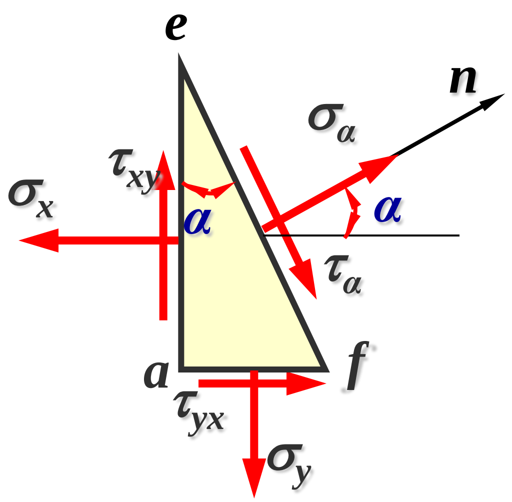
一、斜截面上的应力

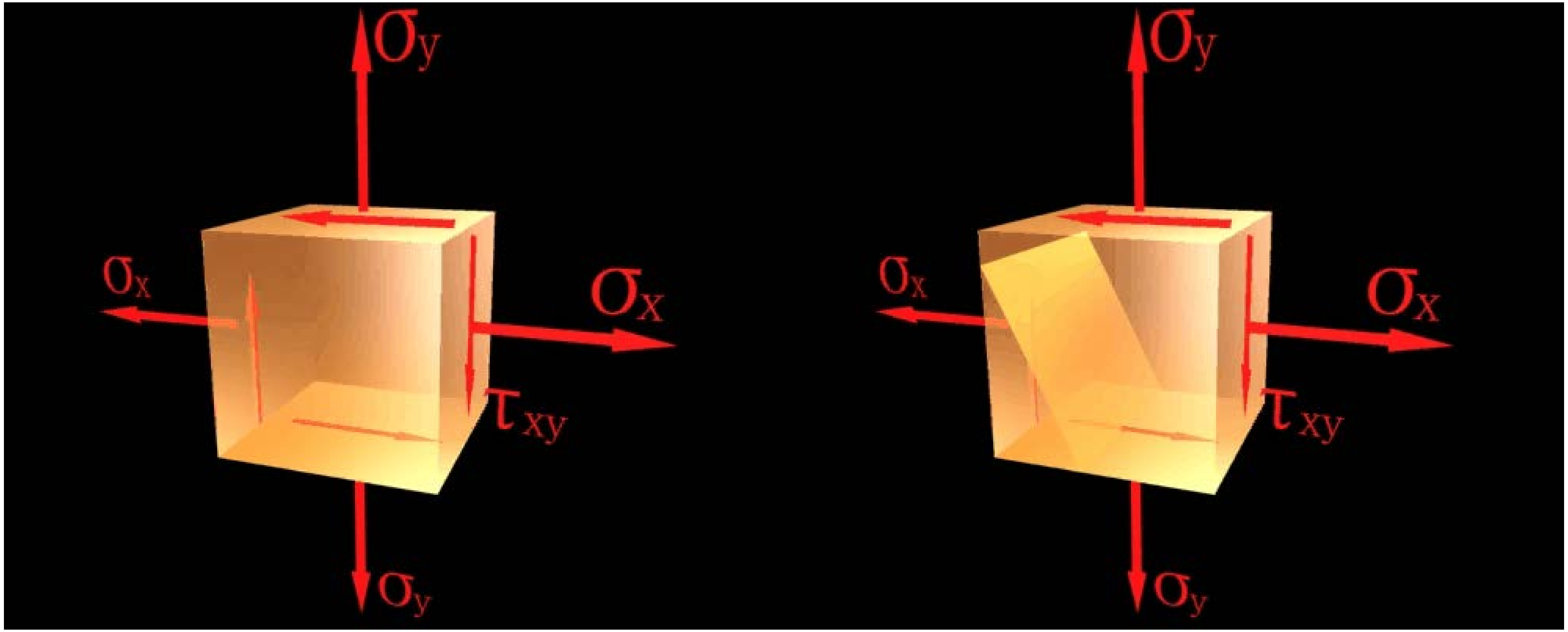
1. 截面法

假想地沿斜截面 *e-f* 将单元体截开，

留下左边部分的单体元 *eaf* 作为研究对象






σy
σx
σx
τxy
σy
σy
σx
σx
τxy
σy

2. 符号的确定

（1）由x轴转到外法线n，

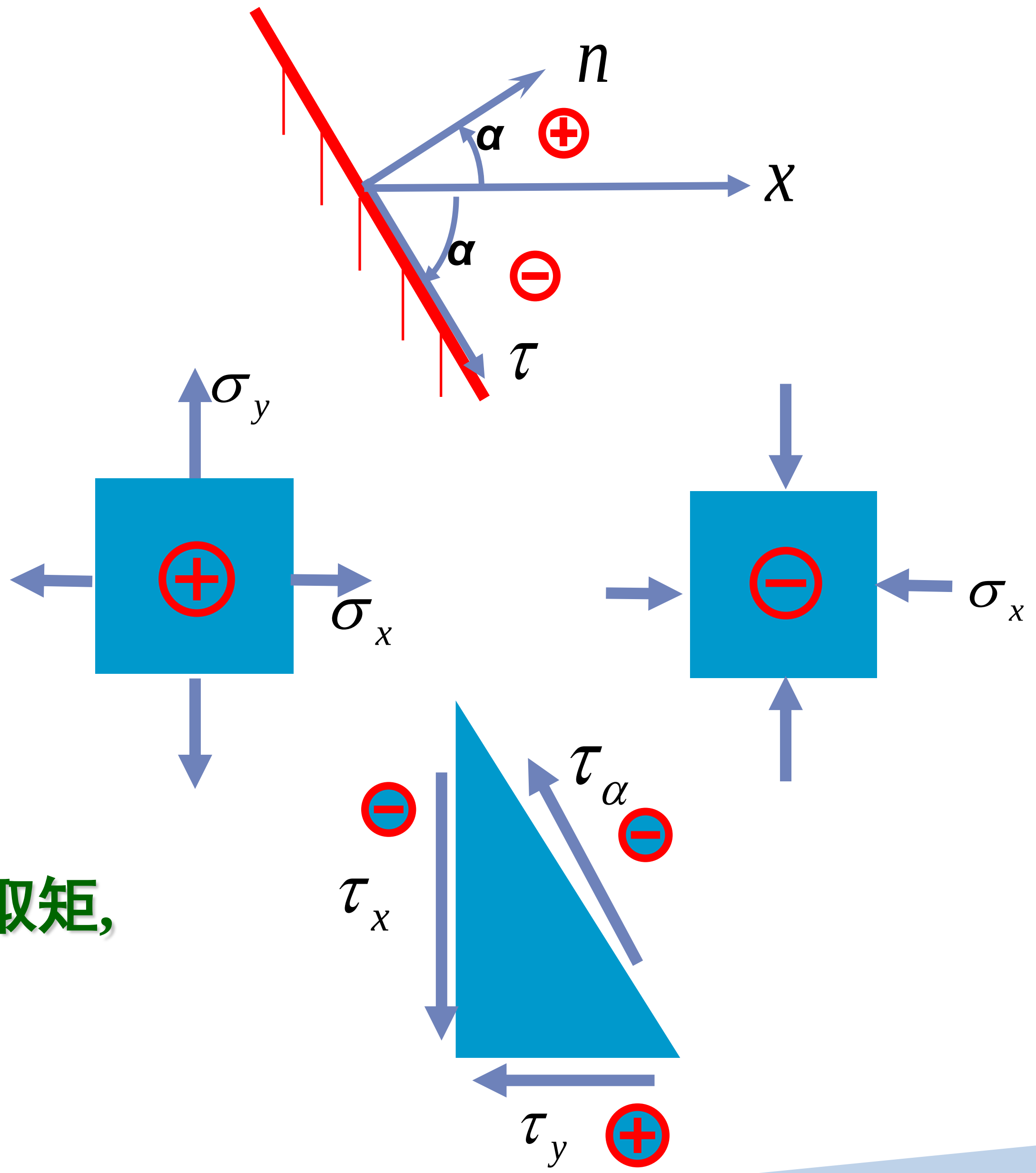
逆时针转向时α为正

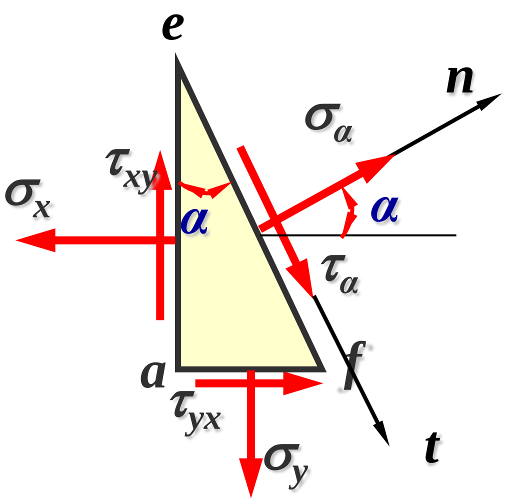
（2）正应力仍规定

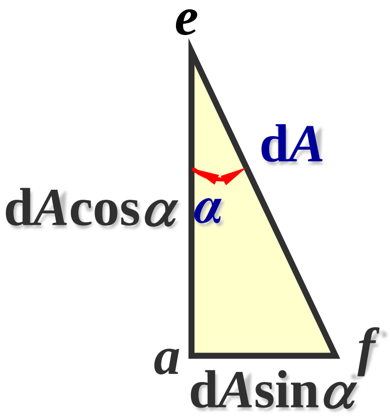
拉应力σ为正

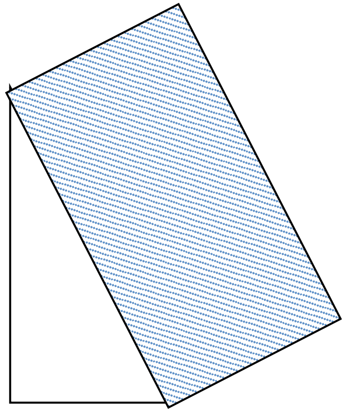
（3）切应力对单元体内任一点取矩，

顺时针转τ为正









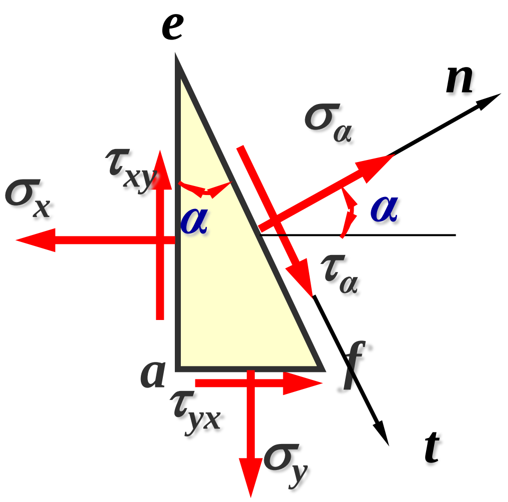
3. 任意斜截面上的应力

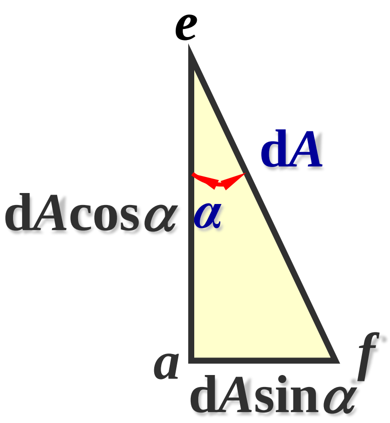
设斜截面的面积为dA , a-e的面积为dAcosα, a-f 的面积为dAsinα

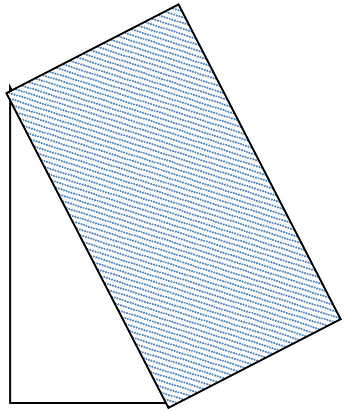
对研究对象列 n和 t 方向的平衡方程得

$$\sum F_n = 0 \quad \sigma_\alpha \mathrm{d}A + (\tau_{xy} \mathrm{d}A\cos\alpha)\sin\alpha - (\sigma_x \mathrm{d}A\cos\alpha)\cos\alpha +$$

$$(\tau_{yx} \mathrm{d}A\sin\alpha)\cos\alpha - (\sigma_y \mathrm{d}A\sin\alpha)\sin\alpha = 0$$





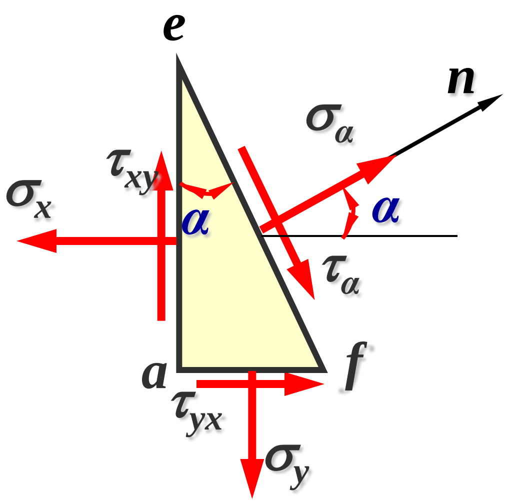


$$\sum F_t = 0 \quad \tau_\alpha \mathrm{d}A - (\tau_{xy}\mathrm{d}A\cos\alpha)\cos\alpha - (\sigma_x \mathrm{d}A\cos\alpha)\sin\alpha +$$

$$(\tau_{yx}\mathrm{d}A\sin\alpha)\sin\alpha + (\sigma_y \mathrm{d}A\sin\alpha)\cos\alpha = 0$$

$$\sum F_n = 0 \quad \sigma_\alpha \mathrm{d}A + (\tau_{xy}\mathrm{d}A\cos\alpha)\sin\alpha - (\sigma_x \mathrm{d}A\cos\alpha)\cos\alpha + (\tau_{yx}\mathrm{d}A\sin\alpha)\cos\alpha - (\sigma_y \mathrm{d}A\sin\alpha)\sin\alpha = 0$$

$$\sum F_t = 0 \quad \tau_\alpha \mathrm{d}A - (\tau_{xy}\mathrm{d}A\cos\alpha)\cos\alpha - (\sigma_x \mathrm{d}A\cos\alpha)\sin\alpha + (\tau_{yx}\mathrm{d}A\sin\alpha)\sin\alpha + (\sigma_y \mathrm{d}A\sin\alpha)\cos\alpha = 0$$



化简以上两个平衡方程最后得

$$\sigma_\alpha = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\alpha - \tau_{xy}\sin 2\alpha$$

$$\tau_\alpha = \frac{\sigma_x - \sigma_y}{2}\sin 2\alpha + \tau_{xy}\cos 2\alpha$$

二、最大正应力及方位

$$\sigma_\alpha = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\alpha - \tau_{xy}\sin 2\alpha$$

$$\tau_\alpha = \frac{\sigma_x - \sigma_y}{2}\sin 2\alpha + \tau_{xy}\cos 2\alpha$$

1. 最大正应力的方位

令 $$\frac{\mathrm{d}\sigma_\alpha}{\mathrm{d}\alpha} = -2\left[\frac{\sigma_x - \sigma_y}{2}\sin 2\alpha + \tau_{xy}\cos 2\alpha\right] = 0$$

$$\tan 2\alpha_0 = -\frac{2\tau_{xy}}{\sigma_x - \sigma_y} \longrightarrow \begin{cases} \alpha_0 \\ \alpha_0 + 90^\circ \end{cases}$$

α_0 和 α_0+90° 确定两个互相垂直的平面, 一个是最大正应力所在的平面, 另一个是最小正应力所在的平面.

2. 最大正应力

将 α_0和 α_0+90° 代入公式

$$\sigma_\alpha = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\alpha - \tau_{xy}\sin 2\alpha$$

得到$\sigma_{\max}$和$\sigma_{\min}$

$$\left\{\begin{matrix}\sigma_{\max} \\ \sigma_{\min}\end{matrix}\right. = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$\tau_{\alpha_0}=0$ ∴极值正应力就是主应力！

若约定$\sigma_x > \sigma_y$, 则 $|\alpha_0|$较小的确定$\sigma_{\max}$所在的平面

三、最大切应力及方位

$$\sigma_\alpha = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\alpha - \tau_{xy}\sin 2\alpha$$

$$\tau_\alpha = \frac{\sigma_x - \sigma_y}{2}\sin 2\alpha + \tau_{xy}\cos 2\alpha$$

1. 最大切应力的方位

令 $$\frac{\mathrm{d}\tau_\alpha}{\mathrm{d}\alpha} = 2[\frac{\sigma_x - \sigma_y}{2}\cos 2\alpha - \tau_{xy}\sin 2\alpha] = 0$$

$$\tan 2\alpha_1 = \frac{\sigma_x - \sigma_y}{2\tau_{xy}} \longrightarrow \begin{cases} \alpha_1 \\ \alpha_1 + 90^\circ \end{cases}$$

α_1 和 α_1+90° 确定两个互相垂直的平面，一个是最大切应力所在的平面，另一个是最小切应力所在的平面。

2. 最大切应力

将α_1和 α_1+90° 代入公式

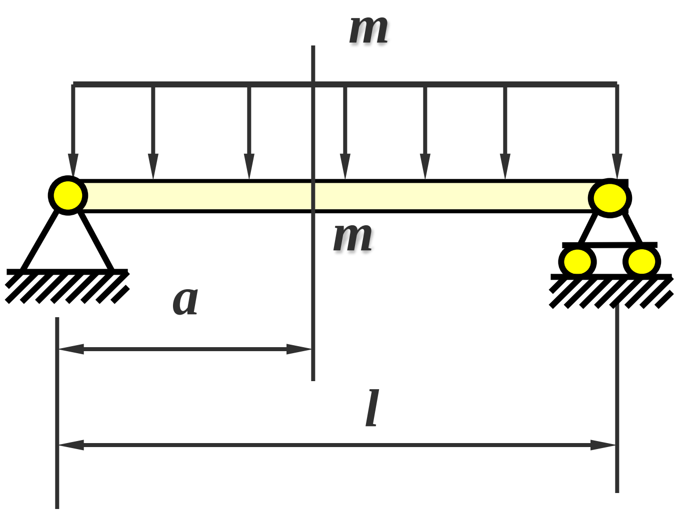
$$\tau_\alpha = \frac{\sigma_x - \sigma_y}{2}\sin 2\alpha + \tau_{xy}\cos 2\alpha$$

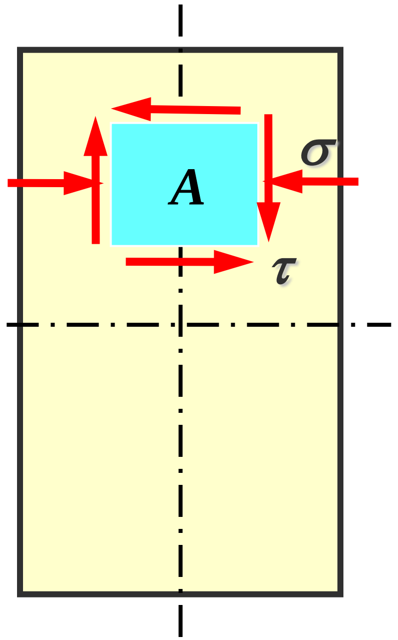
得到τ_{max}和τ_{min} $\left.\begin{cases}\tau_{max} \\ \tau_{min}\end{cases}\right. = \pm\sqrt{(\frac{\sigma_x - \sigma_y}{2})^2 + \tau_{xy}^2}$

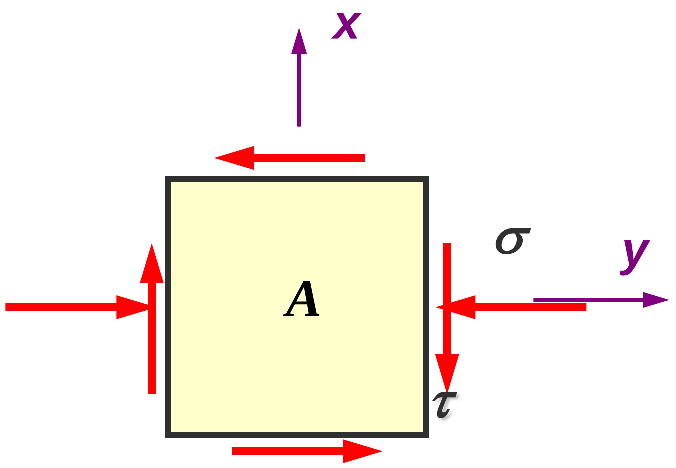
比较 $\tan 2\alpha_0 = -\frac{2\tau_{xy}}{\sigma_x - \sigma_y}$ 和 $\tan 2\alpha_1 = \frac{\sigma_x - \sigma_y}{2\tau_{xy}}$

可见 $\tan 2\alpha_0 = -\frac{1}{\tan 2\alpha_1}$ $\quad 2\alpha_1 = 2\alpha_0 + \frac{\pi}{2}$，$\alpha_1 = \alpha_0 + \frac{\pi}{4}$

例题　简支梁如图所示. 已知 m-m 截面上A点的弯曲正应力和切应力分别为σ =−70MPa, τ=50MPa.确定A点的主应力及主平面的方位。



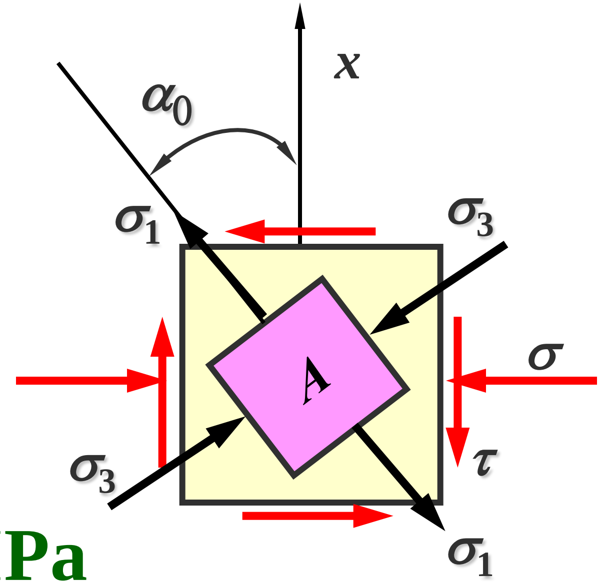




解： 把从A点处截取的单元体（原始单元体）放大如图

$$\sigma_x = 0, \quad \sigma_y = -70, \quad \tau_{xy} = -50$$

$$\tan 2\alpha_0 = -\frac{2\tau_{xy}}{\sigma_x - \sigma_y} = -\frac{2\times(-50)}{0-(-70)} = 1.429$$



$$\alpha_0 = \begin{matrix} 27.5^\circ \\ 117.5^\circ \end{matrix}$$

$$\left\{\begin{matrix} \sigma_{\max} \\ \sigma_{\min} \end{matrix}\right. = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \left\{\begin{matrix} 26\text{MPa} \\ -96\text{MPa} \end{matrix}\right.$$

$$\sigma_1 = 26\text{MPa}, \quad \sigma_2 = 0, \quad \sigma_3 = -96\text{MPa}$$

本讲小结

1 斜截面上的应力

2 最大正应力及方位

3 最大切应力及方位